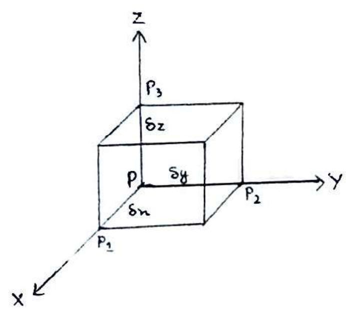
• **Nature of Strains :-**

Strain maybe defined as a non-dimensional deformation which measures the change of relative positions of the parts of body under any cause. Strain is defined into the following two types.

(i) Normal Strain :- The ratio of the change in length to the original length of a linear element is known as the normal strain.

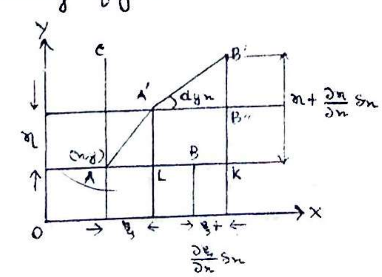
(ii) Shearing Strain ;- It is measured in terms of the change in the angle between two linear elements from the unstrained state to the strained state.



Now we investigate the state of strain at a point P.

We consider a infinitesimal parallelopiped with edges $PP_1 = \delta x$, $PP_2 = \delta y$ $PP_3 = \delta z$. During the deformation of the body it will displace and deform. The length of the edges will get elongated or may be contracted and initially the right angles between the faces will change. To estimate the required deformation at the point P we must evaluate the elongations i.e normal strain of the edges $\delta x, \delta y, \delta z$ of the given parallelopiped and the distortions of the angle $\angle P_1PP_2$ $\angle P_1PP_3$ $\angle P_2PP_3$ for shearing strains. For this we consider the projections of the parallelopiped on the co-ordinate planes

Let us take the projection of the elementary parallelopiped $PP_1P_2P_3$ on the xy-plane as shown in the following figure.



First we find the strain in a two dimensional space. Before strain the lengths of the edges are $AB = \delta x$, $AC = \delta y$, and $\angle BAC = 90°$. After very small strain let AB [illegible] the position A'B'. Let the co-ordinates of A be (x, y) and these of A' be $(x+\xi, y+\eta)$. From the figure it is clear that the displacement of A along x-axis is ξ i.e. $AL = \xi$ hence the corresponding displacement of the point B is BK and is given by $\xi + \frac{\partial \xi}{\partial x}\delta x$

Again the displacement of the point A along the y-axis is η and the displacement of the ~~pom~~ point B along the y-axis is $\eta + \frac{\partial \eta}{\partial x}\delta x$

∴ The normal strain component along the x-direction is given by...

$$(\epsilon_{xx})_s = \frac{A'B' - AB}{AB} \quad \text{--- (i)}$$

where the subscript $(\;)_s$ is used to indicate the quantity being obtained is related to the elastic solid body

Now $A'B' = A'B''$ [since $\angle B'A'B''$ is negligibly small]

$$A'B' = AB + BK - AL$$

$$= \delta x + \xi + \frac{\partial \xi}{\partial x}\delta x - \xi = \delta x + \frac{\partial \xi}{\partial x}\delta x \quad \text{--- (ii)}$$

$$(\epsilon_{xx})_s = \frac{\delta x + \frac{\partial \xi}{\partial x}\delta x - \delta x}{\delta x} = \frac{\partial \xi}{\partial x} \quad \text{(iii)}$$

similarly we can get the normal strain components in the y-direction is $(\epsilon_{yy})_s = \frac{\partial \eta}{\partial y}$ ... (iv)

The shearing strain $(\gamma_{xy})_s$ at the point A is a change of the angle between AB and AC. The angle of rotation α_{yx} of the edge AB is given by....

$$\alpha_{yx} \approx \tan \alpha_{yx} \text{ [taking 1st order approximation]}$$

$$= \frac{B'B''}{A'B''}$$

$$= \frac{B'K - B''K}{A'B''}$$

$$= \frac{\eta + \frac{\partial \eta}{\partial x}\delta x - \eta}{\delta x + \frac{\partial \xi}{\partial x}\delta x}$$

$$= \frac{\frac{\partial \eta}{\partial x}}{1 + \frac{\partial \xi}{\partial x}} = \frac{\partial \eta}{\partial x}$$ (since $\frac{\partial \xi}{\partial x}$ is negligibly small compare to unit)

similarly considering the angle of rotation of the edge AC is given by $\alpha_{xy} = \frac{\partial \xi}{\partial y}$

Thus the shearing strain at the point A is ...

$$(\gamma_{xy})_s = d_{yx} + d_{xy}$$
$$= \frac{\partial \eta}{\partial x} + \frac{\partial \xi}{\partial y} \quad \cdots (v)$$

similarly, we can obtain the expressions of normal strains and shearing strains in the other two co-ordinate planes. If ξ, η, ζ be the components of displacements at P in 3-dimensional case, then the normal strains are ...

$$\left.\begin{aligned}(\epsilon_{xx})_s &= \frac{\partial \xi}{\partial x} \\ (\epsilon_{yy})_s &= \frac{\partial \eta}{\partial y} \\ (\epsilon_{zz})_s &= \frac{\partial \zeta}{\partial z}\end{aligned}\right\} \cdots (vi)$$

and the components of shearing strain are ...

$$\left.\begin{aligned}(\gamma_{xy})_s &= \frac{\partial \eta}{\partial x} + \frac{\partial \xi}{\partial y} = (\gamma_{yx})_s \\ (\gamma_{yz})_s &= \frac{\partial \zeta}{\partial y} + \frac{\partial \eta}{\partial z} = (\gamma_{zy})_s \\ (\gamma_{zx})_s &= \frac{\partial \xi}{\partial z} + \frac{\partial \zeta}{\partial x} = (\gamma_{xz})_s\end{aligned}\right\} \cdots (vii)$$

In elasticity we deal with actual displacement in a unit length.

On the other hand, we are interested in a rate of strain while studying fluid dynamics.

Let u, v, w be the velocity of a fluid particle along x, y, z directions respectively. then the rate of strain in fluid dynamics corresponding to (vi) and (vii) are defined as ...

$$\left.\begin{aligned}\epsilon_{xx} &= \frac{\partial}{\partial t}\left(\frac{\partial \xi}{\partial x}\right) \\ &= \frac{\partial}{\partial x}\left(\frac{\partial \xi}{\partial t}\right) = \frac{\partial u}{\partial x} \\ \text{similarly} \cdots \quad \epsilon_{yy} &= \frac{\partial v}{\partial y}, \quad \epsilon_{zz} = \frac{\partial w}{\partial z}\end{aligned}\right\} \cdots (viii)$$

$$\left.\begin{aligned}\therefore \gamma_{xy} &= \frac{\partial}{\partial t}\left(\frac{\partial \eta}{\partial x} + \frac{\partial \xi}{\partial y}\right) \\ &= \frac{\partial}{\partial x}\left(\frac{\partial \eta}{\partial t}\right) + \frac{\partial}{\partial y}\left(\frac{\partial \xi}{\partial t}\right) \\ &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \\ \text{similarly} \cdots \quad \gamma_{yz} &= \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \quad \cdots \quad \gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\end{aligned}\right\} \cdots (ix)$$

Sometimes we introduce the shear angles in formula (viii) and (ix) as.

$$\epsilon_{xy} = \frac{1}{2}\gamma_{xy} \qquad \epsilon_{yz} = \frac{1}{2}\gamma_{yz} \qquad \epsilon_{zx} = \frac{1}{2}\gamma_{zx} \quad \text{---} \quad (x)$$

Thus the rates of strain components for a fluid particle may be rewritten as.

$$\epsilon_{xx} = \frac{\partial u}{\partial x}, \quad \epsilon_{yy} = \frac{\partial v}{\partial y}, \quad \epsilon_{zz} = \frac{\partial w}{\partial z}$$

$$\epsilon_{xy} = \frac{1}{2}\gamma_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \epsilon_{yx}$$

$$\epsilon_{yz} = \frac{1}{2}\gamma_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) = \epsilon_{zy}$$

$$\epsilon_{zx} = \frac{1}{2}\gamma_{zx} = \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) = \epsilon_{xz} \quad \text{---} \quad (xi)$$

The above nine quantities ϵ_{ij} $(i, j = x, y, z)$ may be arranged as follows:-

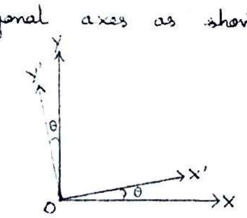
$$\epsilon_{ij} = \begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{yx} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_{zz} \end{bmatrix}$$

The above mentioned 9-quantities constitute the components of the rate of strain tensor of order 2. Since the strains have the nature of a change in displacement in a given unit length in a given direction, the rate of strain is clearly a tensor of order 2. We have seen that $\epsilon_{xy} = \epsilon_{yx}$, $\epsilon_{yz} = \epsilon_{zy}$, $\epsilon_{zx} = \epsilon_{xz}$. It follows that the rate of strain tensor is a symmetric tensor.

- **Transformation of the rate of strain components :-**

- **Case - I :- Two - dimensional rate of strain components :-**

Let the two dimensional rates of strain components be ϵ_{xx}, ϵ_{yy}, ϵ_{xy} at O w.r.t the co-ordinate axes OX, OY known. Let OX' and OY' be the another set of orthogonal axes as shown in the following figure.



We have to find the rates of strain components wrt the new axes OX' and OY' i.e We have to find $\epsilon_{x'x'}$, $\epsilon_{y'y'}$, $\epsilon_{x'y'}$. The dcs of one set of axes wrt the other are shown in the following table.

	OX	OY
OX'	$l_1 = \cos\theta$	$m_1 = \sin\theta$
OY'	$l_2 = -\sin\theta$	$m_2 = \cos\theta$

Let the velocity components be u', v' along OX' and OY' respectively. Thus we have

$$\left.\begin{aligned} x &= l_1 x' + l_2 y' \\ y &= m_1 x' + m_2 y' \end{aligned}\right\} \quad (i)$$

$$\left.\begin{aligned} u' &= l_1 u + m_1 v \\ v' &= l_2 u + m_2 v \end{aligned}\right\} \quad (ii)$$

using (i) and (ii) the rates of strain components $\epsilon_{x'x'}$, $\epsilon_{y'y'}$, $\epsilon_{x'y'}$ in the new co-ordinate system OX', OY' are given by

$$\epsilon_{x'x'} = \frac{\partial u'}{\partial x'} = \frac{\partial u'}{\partial x}\frac{\partial x}{\partial x'} + \frac{\partial u'}{\partial y}\cdot\frac{\partial y}{\partial x'}$$

$$= \left(l_1 \frac{\partial u}{\partial x} + m_1 \frac{\partial v}{\partial x}\right) l_1 + \left(l_1 \frac{\partial u}{\partial y} + m_1 \frac{\partial v}{\partial y}\right) m_1$$

$$= l_1^2 \frac{\partial u}{\partial x} + m_1^2 \frac{\partial v}{\partial y} + l_1 m_1 \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$$

$$= l_1^2 \epsilon_{xx} + m_1^2 \epsilon_{yy} + 2 l_1 m_1 \epsilon_{xy} \quad \text{--- (iii)}$$

similarly $\quad \epsilon_{y'y'} = l_2^2 \epsilon_{xx} + m_2^2 \epsilon_{yy} + 2 l_2 m_2 \epsilon_{xy} \quad \text{--- (iv)}$

$$\epsilon_{x'y'} = \frac{1}{2}\left(\frac{\partial v'}{\partial x'} + \frac{\partial u'}{\partial y'}\right)$$

$$= \frac{1}{2}\left\{\frac{\partial v'}{\partial x}\cdot\frac{\partial x}{\partial x'} + \frac{\partial v'}{\partial y}\cdot\frac{\partial y}{\partial x'} + \frac{\partial u'}{\partial x}\cdot\frac{\partial x}{\partial y'} + \frac{\partial u'}{\partial y}\cdot\frac{\partial y}{\partial y'}\right\}$$

$$= \frac{1}{2}\left\{\left(l_2 \frac{\partial u}{\partial x} + m_2 \frac{\partial v}{\partial x}\right) l_1 + \left(l_2 \frac{\partial u}{\partial y} + m_2 \frac{\partial v}{\partial y}\right)\cdot m_1 + \left(l_1 \frac{\partial u}{\partial x} + m_1 \frac{\partial v}{\partial x}\right) l_2\right.$$

$$\left. + \left(l_1 \frac{\partial u}{\partial y} + m_1 \frac{\partial v}{\partial y}\right)\cdot m_2\right\}.$$

$$= \cancel{\tfrac{1}{2}}\; l_1 l_2 \epsilon_{xx} + m_1 m_2 \epsilon_{yy} + (l_1 m_2 + l_2 m_1)\epsilon_{xy} \quad \text{--- (v)}$$

Now (iii), (iv) and (iv) may be written as

$$\epsilon_{x'x'} = \cos^2\theta\, \epsilon_{xx} + \sin^2\theta\, \epsilon_{yy} + 2\sin\theta\cos\theta\, \epsilon_{xy}$$

$$= \frac{1}{2}(\epsilon_{xx} + \epsilon_{yy}) + \frac{1}{2}(\epsilon_{xx} - \epsilon_{yy})\cos 2\theta + \epsilon_{xy}\sin 2\theta$$

$$\epsilon_{y'y'} = \frac{1}{2}(\epsilon_{xx} + \epsilon_{yy}) - \frac{1}{2}(\epsilon_{xx} - \epsilon_{yy})\cos 2\theta - \epsilon_{xy}\sin 2\theta$$

$$\epsilon_{x'y'} = \cancel{\epsilon_{yy} - \frac{1}{2}(\epsilon_{xx} - \epsilon_{yy}) + \epsilon_{xy}\cos 2\theta}$$

$$= -\frac{1}{2}(\epsilon_{xx} - \epsilon_{yy})\sin 2\theta + \epsilon_{xy}\cos 2\theta$$

- Invarients of rate of strain for two dimensional case :-

Since the rate of strain is a tensor of order 2, there must ~~be~~ exist at least two invariants of the rate of strains. The two ~~basic~~ invariants for the two dimensional strain components are given by :-

$$\epsilon_{x'x'} + \epsilon_{y'y'} = \epsilon_{xx} + \epsilon_{yy}$$

$$\epsilon_{x'x'}\,\epsilon_{y'y'} - \epsilon_{x'y'}^2 = \frac{1}{4}(\epsilon_{xx}+\epsilon_{yy})^2 - \frac{1}{4}\cancel{(\epsilon_{xx}^2-\epsilon_{yy}^2)\cos 2\theta}$$

$$-\frac{1}{2}\cancel{(\epsilon_{xx}+\epsilon_{yy})\epsilon_{xy}\sin 2\theta} + \frac{1}{4}\cancel{(\epsilon_{xx}^2-\epsilon_{yy}^2)\cos 2\theta}$$

$$-\frac{1}{4}(\epsilon_{xx}-\epsilon_{yy})^2\cos^2 2\theta - \frac{1}{2}\cancel{(\epsilon_{xx}-\epsilon_{yy})\epsilon_{xy}\sin 2\theta\cos 2\theta}$$

$$+\frac{1}{2}\cancel{(\epsilon_{xx}+\epsilon_{yy})\epsilon_{xy}\sin 2\theta} - \frac{1}{2}\cancel{(\epsilon_{xx}-\epsilon_{yy})\epsilon_{xy}\cos 2\theta\sin 2\theta}$$

$$-\cancel{\epsilon^2} - \epsilon_{xy}^2\sin^2 2\theta\ \cancel{\mp\frac{1}{4}(\epsilon_{xx}-\epsilon_{yy})^2 + (\epsilon_{xx}-\epsilon_{yy})\epsilon_{xy}}$$

$$\cancel{\cos 2\theta - \epsilon_{xy}^2\cos^2 2\theta} - \frac{1}{4}(\epsilon_{xx}-\epsilon_{yy})^2\sin^2 2\theta$$

$$+(\epsilon_{xx}-\epsilon_{yy})\cdot\cancel{\epsilon_{xy}\cos 2\theta\sin 2\theta} - \epsilon_{xy}^2\cos^2 2\theta$$

$$= \frac{1}{4}(\epsilon_{xx}+\epsilon_{yy})^2 - \frac{1}{4}(\epsilon_{xx}-\epsilon_{yy})^2 - \epsilon_{xy}^2$$

$$= \frac{1}{4}\cdot 4\,\epsilon_{xx}\cdot\epsilon_{yy} - \epsilon_{xy}^2$$

$$= \epsilon_{xx}\,\epsilon_{yy} - \epsilon_{xy}^2$$

Case - II :-

- Three - dimensional rate of strain components :-

Let the three dimensional rates of strain components $\epsilon_{xx}, \epsilon_{yy}, \epsilon_{zz}, \epsilon_{xy}, \epsilon_{yz}, \epsilon_{zx}$ at O wrt the co-ordinate axes OX, OY, OZ be known. Let OX', OY', OZ' be the another set of orthogonal axes. We have to find the expressions of the components $\epsilon_{x'x'}, \epsilon_{y'y'}, \epsilon_{z'z'}, \epsilon_{x'y'}, \epsilon_{y'z'}, \epsilon_{z'x'}$ wrt the new axes OX', OY', OZ'. Also let the dcs of each axis of the one system wrt the other are shown in the following table.

	OX	OY	OZ
OX'	l_1	m_1	n_1
OY'	l_2	m_2	n_2
OZ'	l_3	m_3	n_3

The table means that (l_1, m_1, n_1) are the dcs of OX' wrt OX OY OZ and that (l_1, l_2, l_3) are the dcs of OX, wrt OX', OY' OZ'. This dcs satisfy the following relations.

$$l_1^2 + m_1^2 + n_1^2 = 1 \qquad l_1^2 + l_2^2 + l_3^2 = 1$$
$$l_2^2 + m_2^2 + n_2^2 = 1 \qquad m_1^2 + m_2^2 + m_3^2 = 1 \quad \cdots (i)$$
$$l_3^2 + m_3^2 + n_3^2 = 1 \qquad n_1^2 + n_2^2 + n_3^2 = 1$$

and

$$l_1 m_1 + l_2 m_2 + l_3 m_3 = 0 \qquad l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$$
$$m_1 n_1 + m_2 n_2 + m_3 n_3 = 0 \qquad l_2 l_3 + m_2 m_3 + n_2 n_3 = 0 \quad \cdots (ii)$$
$$n_1 l_1 + n_2 l_2 + n_3 l_3 = 0 \qquad l_3 l_1 + m_3 m_1 + n_3 n_1 = 0$$

Let u', v', w' be the velocity components along OX', OY', OZ' respectively. Thus we have

$$\left.\begin{aligned} x &= l_1 x' + l_2 y' + l_3 z' \\ y &= m_1 x' + m_2 y' + m_3 z' \\ z &= n_1 x' + n_2 y' + n_3 z' \end{aligned}\right\} \cdots (iii)$$

$$\left.\begin{aligned} u' &= l_1 u + m_1 v + n_1 w \\ v' &= l_2 u + m_2 v + n_2 w \\ w' &= l_3 u + m_3 v + n_3 w \end{aligned}\right\} \cdots (iv)$$

using (iii) and (iv) the rates of strain in the new co-ordinate system are given by

$$\epsilon_{x'x'} = \frac{\partial u'}{\partial x'} = \frac{\partial u'}{\partial x}\cdot\frac{\partial x}{\partial x'} + \frac{\partial u'}{\partial y}\cdot\frac{\partial y}{\partial x'} + \frac{\partial u'}{\partial z}\cdot\frac{\partial z}{\partial x'}$$

$$= \left(l_1 \frac{\partial u}{\partial x} + m_1 \frac{\partial v}{\partial x} + n_1 \frac{\partial w}{\partial x}\right) l_1 + \left(l_1 \frac{\partial u}{\partial y} + m_1 \frac{\partial v}{\partial y} + n_1 \frac{\partial w}{\partial y}\right) m_1$$
$$+ \left(l_1 \frac{\partial u}{\partial z} + m_1 \frac{\partial v}{\partial z} + n_1 \frac{\partial w}{\partial z}\right) n_1$$

$$= l_1^2 \epsilon_{xx} + m_1^2 \epsilon_{yy} + n_1^2 \epsilon_{zz} + 2 l_1 m_1 \epsilon_{xy} + 2 m_1 n_1 \epsilon_{yz} + 2 n_1 l_1 \epsilon_{zx} \cdots (v)$$

similarly ...

$$\epsilon_{y'y'} = l_2^2 \epsilon_{xx} + m_2^2 \epsilon_{yy} + n_2^2 \epsilon_{zz} + 2 l_2 m_2 \epsilon_{xy} + 2 m_2 n_2 \epsilon_{yz} + 2 n_2 l_2 \epsilon_{zx} \cdots (vi)$$

$$\epsilon_{z'z'} = l_3^2 \epsilon_{xx} + m_3^2 \epsilon_{yy} + n_3^2 \epsilon_{zz} + 2 l_3 m_3 \epsilon_{xy} + 2 m_3 n_3 \epsilon_{yz} + 2 n_3 l_3 \epsilon_{zx}$$

(vii)

$$\epsilon_{x'y'} = \frac{1}{2}\left(\frac{\partial v'}{\partial x'} + \frac{\partial u'}{\partial y'}\right)$$

$$= \frac{1}{2}\left(\frac{\partial v'}{\partial x}\cdot\frac{\partial x}{\partial y}\cdot\frac{\partial x}{\partial x'} + \cancel{\frac{\partial v'}{\partial y}}\cdot\frac{\partial y}{\partial x'} + \frac{\partial u'}{\partial x}\cdot\frac{\partial x}{\partial y'} + \frac{\partial u'}{\partial y}\cdot\frac{\partial y}{\partial y'}\right)$$

$$\frac{1}{2}\left\{\left(l_2\frac{\partial u}{\partial x} + m_2\frac{\partial v}{\partial x} + n_2\frac{\partial w}{\partial x}\right)l_1 + \left(l_2\frac{\partial u}{\partial y} + m_2\frac{\partial v}{\partial y} + n_2\frac{\partial w}{\partial y}\right)m_1\right.$$

$$\left. + \left(l_1\frac{\partial u}{\partial x} + m_1\frac{\partial v}{\partial x} + n_1\frac{\partial w}{\partial x}\right)l_2 + \left(l_1\frac{\partial u}{\partial y} + m_1\frac{\partial v}{\partial y} + n_1\frac{\partial w}{\partial y}\right)m_2\right\}$$

$$\cancel{= l_1 l_2 \epsilon_{xx} + m_1 m_2 \epsilon_{yy} +}$$

$$= \frac{1}{2}\left(\frac{\partial v'}{\partial x}\cdot\frac{\partial x}{\partial x'} + \frac{\partial v'}{\partial y}\cdot\frac{\partial y}{\partial x'} + \frac{\partial v'}{\partial z}\cdot\frac{\partial z}{\partial x'} + \frac{\partial u'}{\partial x}\cdot\frac{\partial x}{\partial y'} + \frac{\partial u'}{\partial y}\cdot\frac{\partial y}{\partial y'} + \frac{\partial u'}{\partial z}\cdot\frac{\partial z}{\partial y'}\right)$$

$$= \frac{1}{2}\left\{\left(l_2\frac{\partial u}{\partial x} + m_2\frac{\partial v}{\partial x} + n_2\frac{\partial w}{\partial x}\right)l_1 + \left(l_2\frac{\partial u}{\partial y} + m_2\frac{\partial v}{\partial y} + n_2\frac{\partial w}{\partial y}\right)m_1 +\right.$$

$$\left(l_2\frac{\partial u}{\partial z} + m_2\frac{\partial v}{\partial z} + n_2\frac{\partial w}{\partial z}\right)n_1 + \left(l_1\frac{\partial u}{\partial x} + m_1\frac{\partial v}{\partial x} + n_1\frac{\partial w}{\partial x}\right)l_2 +$$

$$\left.\left(l_1\frac{\partial u}{\partial y} + m_1\frac{\partial v}{\partial y} + n_1\frac{\partial w}{\partial y}\right)m_2 + \left(l_1\frac{\partial u}{\partial z} + m_1\frac{\partial v}{\partial z} + n_1\frac{\partial w}{\partial z}\right)n_2\right\}$$

$$= l_1 l_2 \epsilon_{xx} + m_1 m_2 \epsilon_{yy} + n_1 n_2 \epsilon_{zz} + (l_1 m_2 + l_2 m_1)\epsilon_{xy} + (m_1 n_2 + m_2 n_1)\epsilon_{yz}$$

$$+ (n_1 l_2 + n_2 l_1)\epsilon_{zx}$$

Similarly

$$\epsilon_{y'z'} = l_2 l_3 \epsilon_{xx} + m_2 m_3 \epsilon_{yy} + n_2 n_3 \epsilon_{zz} + (l_2 m_3 + l_3 m_2)\epsilon_{xy}$$

$$+ (m_2 n_3 + m_3 n_2)\epsilon_{yz} + (n_2 l_3 + n_3 l_2)\epsilon_{zx}$$

$$\epsilon_{z'x'} = l_3 l_1 \epsilon_{xx} + m_3 m_1 \epsilon_{yy} + n_3 n_1 \epsilon_{zz} + (l_3 m_1 + l_1 m_3)\epsilon_{xy}$$

$$+ (m_3 n_1 + m_1 n_3)\epsilon_{yz} + (n_3 l_1 + n_1 l_3)\epsilon_{zx}$$

• Invariants of the rate of strain for three dimensional case :-

The invariants in the 3-dimensional case is given by ---

(i) $\epsilon_{x'x'} + \epsilon_{y'y'} + \epsilon_{z'z'} = \epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz}$

(ii) $\epsilon_{x'x'}\epsilon_{y'y'} + \epsilon_{y'y'}\epsilon_{z'z'} + \epsilon_{z'z'}\epsilon_{x'x'} - (\epsilon_{x'y'}^2 + \epsilon_{y'z'}^2 + \epsilon_{z'x'}^2)$

$= \epsilon_{xx}\epsilon_{yy} + \epsilon_{yy}\epsilon_{zz} + \epsilon_{zz}\epsilon_{xx} - (\epsilon_{xy}^2 + \epsilon_{yz}^2 + \epsilon_{zx}^2)$

(iii) $\epsilon_{x'x'}\epsilon_{y'y'}\epsilon_{z'z'} + (\epsilon_{x'y'}\epsilon_{y'z'}\epsilon_{z'x'} - \epsilon_{x'x'}\epsilon_{y'z'}^2 - \cancel{\epsilon_{z'z'}\epsilon_{x'y'}^2}$

$- \epsilon_{y'y'}\epsilon_{z'x'}^2 - \epsilon_{z'z'}\epsilon_{x'y'}^2) = \epsilon_{xx}\epsilon_{yy}\epsilon_{zz} + (\epsilon_{xy}\epsilon_{yz}\epsilon_{zx} -$

$\epsilon_{xx}\epsilon_{yz}^2 - \epsilon_{yy}\epsilon_{zx}^2 - \epsilon_{zz}\epsilon_{xy}^2)$

• The constitutive equations for a compressible Newtonian viscous fluid :-
(relation between stress and rates of strain):-

⇒ The fluid flow can be analised into the following three parts :-

(i) A rigid body translation at a velocity equal to the fluid velocity at a particular point P.

(ii) Rigid body rotations about the axes through the point P.

(iii) Straining motion of distortion, characterised by the rate of strain.

Since stresses will be generated by either the translation or rigid body rotation, it follows that the stress tensor can be determined by the rate of strain tensor.

To find the relation between the stress and the rates of strain we have made the following assumptions :-

(i) The stress componants may be expressed as a linear function of the rates of strain componants.

(ii) The relations between stress componants and the rates of strain componants must be invariant to a co-ordinate transformation consisting of either a rotation or a mirror reflexion of ~~axes~~ axes.

(iii) The stress componants must reduce to the hydrostatic pressure p when all the gradients of velocity are zero.

With the above assumptions we will find the relation between the stress and the rate of strain componants in two dimensional case.

The relations so obtained will be further extended two three dimensional ~~case~~ case. such relations are called the constitutive equations.

Relation between stress and the rate of strain components in two-dimensional case :-

In view of assumption (i), let us take.

$$\left.\begin{aligned}\sigma_{xx} &= A_1 \epsilon_{xx} + B_1 \epsilon_{yy} + C_1 \gamma_{xy} + D_1 \\ \sigma_{yy} &= A_2 \epsilon_{xx} + B_2 \epsilon_{yy} + C_2 \gamma_{xy} + D_2 \\ \sigma_{xy} &= A_3 \epsilon_{xx} + B_3 \epsilon_{yy} + C_3 \gamma_{xy} + D_3\end{aligned}\right\} \text{--- (1)}$$

where A's, B's, C's, D's are constants to be determined with the help of assumptions (ii) and (iii). Let $\sigma_{x'x'}$, $\sigma_{y'y'}$, $\sigma_{x'y'}$ be the stress components wrt the new axes OX', OY' also let $\epsilon_{x'x'}$, $\epsilon_{y'y'}$, $\gamma_{x'y'}$ be the rate of strain components wrt the new-axes. In view of the assumption (ii) the stress and rate of strain relation must remain unaltered wrt the new co-ordinate system. Thus we get.

$$\left.\begin{aligned}\sigma_{x'x'} &= A_1 \epsilon_{x'x'} + B_1 \epsilon_{y'y'} + C_1 \gamma_{x'y'} + D_1 \\ \sigma_{y'y'} &= A_2 \epsilon_{x'x'} + B_2 \epsilon_{y'y'} + C_2 \gamma_{x'y'} + D_2 \\ \sigma_{x'y'} &= A_3 \epsilon_{x'x'} + B_3 \epsilon_{y'y'} + C_3 \gamma_{x'y'} + D_3\end{aligned}\right\} \text{--- (2)}$$

We know that the relation between the stress-components in old and new co-ordinate axes, then we have..

$$\sigma_{x'x'} = \frac{1}{2}(1+\cos 2\theta)\sigma_{xx} + \frac{1}{2}(1-\cos 2\theta)\sigma_{yy} + \sigma_{xy}\sin 2\theta \text{ ---- (3)}$$

using (1), (3) becomes

$$\sigma_{x'x'} = \left[\frac{A_1}{2}(1+\cos 2\theta) + \frac{A_2}{2}(1-\cos 2\theta) + A_3 \sin 2\theta\right] \cancel{\epsilon_{yy}}\ \epsilon_{xx}$$

$$+ \left[\frac{B_1}{2}(1+\cos 2\theta) + \frac{B_2}{2}(1-\cos 2\theta) + B_3 \sin 2\theta\right] \epsilon_{yy}$$

$$+ \left[\frac{C_1}{2}(1+\cos 2\theta) + \frac{C_2}{2}(1-\cos 2\theta) + C_3 \sin 2\theta\right] \gamma_{xy}$$

$$+ \left[\frac{D_1}{2}(1+\cos 2\theta) + \frac{D_2}{2}(1-\cos 2\theta) + D_3 \sin 2\theta\right] \text{ ..... (4)}$$

Again we have the relations between the rate of strain tensor in old and new co-ordinate axes..

$$\epsilon_{x'x'} = \frac{1}{2}(\epsilon_{xx}+\epsilon_{yy}) + \frac{1}{2}(\epsilon_{xx}-\epsilon_{yy})\cos 2\theta + \frac{1}{2}\gamma_{xy}\sin 2\theta$$

$$\epsilon_{y'y'} = \frac{1}{2}(\epsilon_{xx}+\epsilon_{yy}) - \frac{1}{2}(\epsilon_{xx}-\epsilon_{yy})\cos 2\theta - \frac{1}{2}\gamma_{xy}\sin 2\theta \quad \text{--- (5)}$$

$$\gamma_{x'y'} = (\epsilon_{yy}-\epsilon_{xx})\sin 2\theta + \gamma_{xy}\cos 2\theta$$

substituting the values of $\epsilon_{x'x'}$, $\epsilon_{y'y'}$, $\gamma_{x'y'}$ from (5) in (2) we obtain...

$$\sigma_{x'x'} = \left[\frac{A_1}{2}(1+\cos 2\theta) + \frac{B_1}{2}(1-\cos 2\theta) - C_1\sin 2\theta\right]\epsilon_{xx}$$

$$+\left[\frac{A_1}{2}(1-\cos 2\theta) + \frac{B_1}{2}(1+\cos 2\theta) + C_1\sin 2\theta\right]\epsilon_{yy}$$

$$+\left[\frac{A_1}{2}\sin 2\theta - \frac{B_1}{2}\sin 2\theta + C_1\cos 2\theta\right]\gamma_{xy} + D_1 \quad \text{.... (6)}$$

since (4) and (6) are identical, it follows that the co-efficients of ϵ_{xx}, ϵ_{yy}, γ_{xy} and the constant terms in these equation must be the same $\forall$ values of θ i.e.

$$\frac{A_1}{2}(1+\cos 2\theta) + \frac{A_2}{2}(1-\cos 2\theta) + A_3\sin 2\theta = \frac{A_1}{2}(1+\cos 2\theta) + \frac{B_1}{2}(1-\cos 2\theta) - C_1\sin 2\theta \quad (7)$$

$$\frac{B_1}{2}(1+\cos 2\theta) + \frac{B_2}{2}(1-\cos 2\theta) + B_3\sin 2\theta = \frac{A_1}{2}(1-\cos 2\theta) + \frac{B_1}{2}(1+\cos 2\theta) + C_1\sin 2\theta \quad (8)$$

$$\frac{C_1}{2}(1+\cos 2\theta) + \frac{C_2}{2}(1-\cos 2\theta) + C_3\sin 2\theta = \frac{A_1}{2}\sin 2\theta - \frac{B_1}{2}\sin 2\theta + C_1\cos 2\theta \quad (9)$$

$$\frac{D_1}{2}(1+\cos 2\theta) + \frac{D_2}{2}(1-\cos 2\theta) + D_3\sin 2\theta = D_1 \quad \text{.... (10)}$$

since (7), (8), (9) and (10) are all true $\forall\ \theta$, so equating the co-efficients of $\cos 2\theta$, $\sin 2\theta$ and the constant term we get....

$$\frac{1}{2}(A_1+A_2) = \frac{1}{2}(A_1+B_1) \quad \text{.... (11)}$$

$$\frac{1}{2}(A_1-A_2) = \frac{1}{2}(A_1-B_1) \quad \text{.... (12)}$$

$$A_3 = -C_1 \quad \text{...... (13)}$$

$$\frac{1}{2}(B_1+B_2) = \frac{1}{2}(A_1+B_1) \quad \text{... (14)}$$

$$\frac{1}{2}(B_1-B_2) = \frac{1}{2}(-A_1+B_1) \quad (15)$$

$$B_3 = C_1 \quad \text{..... (16)}$$

$$\frac{1}{2}(C_1+C_2) = 0 \quad \text{.... (17)}$$

$$\frac{1}{2}(C_1-C_2) = C_1 \quad \text{.... (18)}$$

$$C_3 = \frac{1}{2}(A_1-B_1) \quad \text{..... (19)}$$

$$\frac{1}{2}(D_1 + D_2) = D_1 \quad \text{......} \quad (20)$$

$$\frac{1}{2}(D_1 - D_2) = 0 \quad \text{......} \quad (21)$$

$$D_3 = 0 \quad \text{......} \quad (22)$$

From (11) and (12) we get.

$$A_2 = B_1 = B \text{ (say)} \quad \text{......} \quad (23)$$

From (14) and (15) we get...

$$B_2 = A_1 = A \text{ (say)} \quad \text{......} \quad (24)$$

From (17) we get $\quad C_1 = -C_2 \quad$ (25)

From (13) (16) and (25) we get

$$C_1 = -A_3 = B_3 = -C_2 = C \text{ (say)}. \quad (26)$$

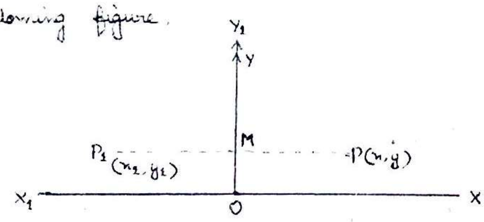
From (20) and (21) we get

$$D_1 = D_2 = D \text{ (say)}. \quad (27)$$

using (22), (23), (24), (26) and (27), the assumed linear relation reduces to...

$$\left.\begin{aligned} \sigma_{xx} &= A\epsilon_{xx} + B\epsilon_{yy} + C\gamma_{xy} + D \\ \sigma_{yy} &= B\epsilon_{xx} + A\epsilon_{yy} - C\gamma_{xy} + D \\ \sigma_{xy} &= C(\epsilon_{yy} - \epsilon_{xx}) + \tfrac{1}{2}(A-B)\gamma_{xy} \end{aligned}\right\} \text{---} \quad (28)$$

Now we proceed to fulfill the 2nd part of the assumption (ii). The proposed relations must be invariant to a new co-ordinate system $OX_1 OY_1$ which is a mirror reflexion of the original system wrt the y-axis as shown in the following figure.



Here $x_1 = -x$ and $y_1 = y$

and $u_1 = -u$ and $v_1 = v$

$\therefore \quad PM = P_1M$

$$\epsilon_{x_1x_1} = \frac{\partial u_1}{\partial x_1} = \frac{\partial u}{\partial x} = \epsilon_{xx}$$

$$\epsilon_{y_1 y_1} = \frac{\partial v_1}{\partial y_1} = \frac{\partial v}{\partial y} = \epsilon_{yy}$$

$$\gamma_{x_1 y_1} = \frac{\partial v_1}{\partial x_1} + \frac{\partial u_1}{\partial y_1} = -\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) = -\gamma_{xy}$$

and $\sigma_{x_1 x_1} = \sigma_{xx}$, $\sigma_{y_1 y_1} = \sigma_{yy}$. $\sigma_{x_1 y_1} = -\sigma_{xy}$

using these values in (28), reduces to.

$$\left.\begin{aligned} \sigma_{x_1 x_1} &= A\,\epsilon_{x_1 x_1} + B\,\epsilon_{y_1 y_1} - C\,\gamma_{x_1 y_1} + D \\ \sigma_{y_1 y_2} &= B\,\epsilon_{x_1 x_1} + A\,\epsilon_{y_1 y_1} + C\,\gamma_{x_1 y_1} + D \\ \sigma_{x_1 y_1} &= C\,(\epsilon_{x_1 x_1} - \epsilon_{y_1 y_1}) + \tfrac{1}{2}(A-B)\,\gamma_{x_1 y_1} \end{aligned}\right\} \quad (29)$$

In view of the form of equation (28) must be the same as that of equation (29).

Accordingly we must choose $C = 0$.

According to the assumption (iii), from equation (28) we must have $D = -p$

Hence equation (28) reduces to ....

$$\left.\begin{aligned} \sigma_{xx} &= A\,\epsilon_{xx} + B\,\epsilon_{yy} - p \\ \sigma_{yy} &= B\,\epsilon_{xx} + A\,\epsilon_{yy} - p \\ \sigma_{xy} &= \tfrac{1}{2}(A-B)\,\gamma_{xy} \end{aligned}\right\} \quad (30)$$

The constant of proportionality $\frac{1}{2}(A-B)$ in the last equation of (30) is taken as μ

$\therefore \frac{1}{2}(A-B) = \mu \Rightarrow A = 2\mu + B$

Then equation (30) reduces to ..

$$\left.\begin{aligned} \sigma_{xx} &= 2\mu\,\epsilon_{xx} + B\,(\epsilon_{xx} + \epsilon_{yy}) - p \\ \sigma_{yy} &= 2\mu\,\epsilon_{yy} + B\,(\epsilon_{xx} + \epsilon_{yy}) - p \\ \sigma_{xy} &= \mu\,\gamma_{xy} = 2\mu\,\epsilon_{xy} \quad \left[\because \tfrac{1}{2}\gamma_{xy} = \epsilon_{xy}\right] \end{aligned}\right\} \quad (31)$$

Extending these the stress and the rate of strain relations in (31) to a three dimensional case, we have.

$$\left.\begin{aligned} \sigma_{xx} &= 2\mu\,\epsilon_{xx} + B\,(\epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz}) - p = 2\mu\,\epsilon_{xx} + B\,\vec{\nabla}\cdot\vec{q} - p \\ \sigma_{yy} &= 2\mu\,\epsilon_{yy} + B\,\vec{\nabla}\cdot\vec{q} - p \\ \sigma_{zz} &= 2\mu\,\epsilon_{zz} + B\,\vec{\nabla}\cdot\vec{q} - p \\ \sigma_{xy} &= 2\mu\,\epsilon_{xy} = 2\mu\cdot\tfrac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) = \sigma_{yx}, \quad \sigma_{yz} = 2\mu\,\epsilon_{yz} = \sigma_{zy} \\ \sigma_{zx} &= 2\mu\,\epsilon_{zx} = \sigma_{xz} \end{aligned}\right\} \quad (32)$$

Here we have

$$\epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \vec{\nabla} \vec{q}$$

$$\vec{q} = u\hat{i} + v\hat{j} + w\hat{k}$$

Adding the first three equations of (32), we get

$$\sigma_{xx} + \sigma_{yy} + \sigma_{zz} = 2\mu \vec{\nabla}.\vec{q} + 3B \vec{\nabla} \vec{q} - 3p \quad (33)$$

Let us assume that the pressure is equal to the mean of the 3-normal stresses $\sigma_{xx}, \sigma_{yy}, \sigma_{zz}$ i.e.

$$\frac{1}{3}(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}) = -p \quad (34)$$

Then (33) reduces to

$$(2\mu + 3B)\vec{\nabla}.\vec{q} = 0$$

Since $\vec{\nabla}.\vec{q} \neq 0$ for compressible fluids, we have $2\mu + B\,3B = 0$

$$\Rightarrow B = -\frac{2\mu}{3}$$

Using $B = -\frac{2\mu}{3}$ in (32) we get the constitutive relations for a compressible fluids.

- Determine the rates of strain and also prove that the first and 2nd invariants for the following velocity components.

$$u = a + by - cz$$

$$v = d - bx + ez$$

$$w = f + cx - ey$$

where a, b, c, d, e, f are arbitary constants

$$\Rightarrow \quad \epsilon_{xx} = \frac{\partial u}{\partial x} = 0$$

$$\epsilon_{yy} = \frac{\partial v}{\partial y} = 0$$

$$\epsilon_{zz} = \frac{\partial w}{\partial z} = 0$$

$$\epsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \frac{1}{2}(b - b) = 0$$

$$\epsilon_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) = \frac{1}{2}(e - e) = 0$$

$$\epsilon_{zx} = \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)$$

$$= \frac{1}{2}(c - c) = 0$$

Now. $\epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz} = 0$

~~$\epsilon_{x'x'} = l_1^2 \epsilon_{xx} + m_1^2 \epsilon_{yy} + 2 l_1 m_1 \epsilon_{xy} = 0$~~

~~$\epsilon_{y'y'} = l_2^2 \epsilon_{xx} + m_2^2 \epsilon_{yy} + 2 l_2 m_2 \epsilon_{xy} = 0$~~

~~$\epsilon_{z'z'} = l_3^2 \epsilon_{xx} + m_3^2 \epsilon_{yy}$~~

$$\epsilon_{x'x'} = l_1^2 \epsilon_{xx} + m_1^2 \epsilon_{yy} + n_1^2 \epsilon_{zz} + 2 l_1 m_1 \epsilon_{xy} + 2 m_1 n_1 \epsilon_{yz} + 2 n_1 l_1 \epsilon_{zx} = 0$$

$$\epsilon_{y'y'} = l_2^2 \epsilon_{xx} + m_2^2 \epsilon_{yy} + n_2^2 \epsilon_{zz} + 2 l_2 m_2 \epsilon_{xy} + 2 m_2 n_2 \epsilon_{yz} + 2 n_2 l_2 \epsilon_{zx} = 0$$

$$\epsilon_{z'z'} = l_3^2 \epsilon_{xx} + m_3^2 \epsilon_{yy} + n_3^2 \epsilon_{zz} + 2 l_3 m_3 \epsilon_{xy} + 2 m_3 n_3 \epsilon_{yz} + 2 n_3 l_3 \epsilon_{zx} = 0$$

$$\epsilon_{x'y'} = l_1 l_2 \epsilon_{xx} + m_1 m_2 \epsilon_{yy} + n_1 n_2 \epsilon_{zz} + (l_1 m_2 + l_2 m_1) \epsilon_{xy} + (m_1 n_2 + n_1 m_2) \epsilon_{yz}$$
$$+ (n_1 l_2 + n_2 l_1) \epsilon_{zx} = 0$$

$$\epsilon_{y'z'} = l_2 l_3 \epsilon_{xx} + m_2 m_3 \epsilon_{yy} + n_2 n_3 \epsilon_{zz} + (l_2 m_3 + l_3 m_2) \epsilon_{xy} + (m_2 n_3 + m_3 n_2) \epsilon_{yz}$$
$$+ (n_2 l_3 + n_3 l_2) \epsilon_{zx} = 0$$

$$\epsilon_{z'x'} = l_3 l_1 \epsilon_{xx} + m_3 m_1 \epsilon_{yy} + n_3 n_1 \epsilon_{zz} + (l_3 m_1 + l_1 m_3) \epsilon_{xy} + (m_3 n_1 + m_1 n_3) \epsilon_{yz}$$
$$+ (n_3 l_1 + n_1 l_3) \epsilon_{zx} = 0$$

Now. $\epsilon_{x'x'} + \epsilon_{y'y'} + \epsilon_{z'z'} = 0 = \epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz}$

and $\epsilon_{x'x'} \epsilon_{y'y'} + \epsilon_{y'y'} \epsilon_{z'z'} + \epsilon_{z'z'} \epsilon_{x'x'} - (\epsilon_{x'y'}^2 + \epsilon_{y'z'}^2 + \epsilon_{z'x'}^2) =$

$= 0 = \epsilon_{xx} \epsilon_{yy} + \epsilon_{yy} \epsilon_{zz} + \epsilon_{zz} \epsilon_{xx} - (\epsilon_{xy}^2 + \epsilon_{yz}^2 + \epsilon_{zx}^2)$

- If a new co-ordinate system (x', y') is obtained from the original co-ordinate system (x, y) by rotation through an angle $30°$, verify the following invariants of the rate of strain for the flow. $u = ay$, $v = 0$.

$$\epsilon_{x'x'} + \epsilon_{y'y'} = \epsilon_{xx} + \epsilon_{yy}$$

$$\epsilon_{x'x'} \epsilon_{y'y'} - \epsilon_{x'y'}^2 = \epsilon_{xx} \epsilon_{yy} - \epsilon_{xy}^2$$

$\Rightarrow$ $\epsilon_{xx} = \frac{\partial u}{\partial x} = 0$

$\epsilon_{yy} = \frac{\partial v}{\partial y} = 0$

~~ϵ_{zz}~~

$\epsilon_{xy} = \frac{1}{2} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)$

$= \frac{1}{2} a$